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Counterterms in type-I supergravities

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Abstract: We compute the one-loop divergences of $D = 10$, $N = 1$ supergravity and of its reduction to $D = 8$. We study the tensor structure of the counterterms appearing in $D = 8$ and $D = 10$ and compare these to expressions previously found in the low energy expansion of string theory. The infinities have the primitive Yang-Mills tree amplitude as a common factor.

Keywords: Superstrings and Heterotic Strings, Extended Supersymmetry, Supergravity Models.

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Contents

1. Introduction

Highly extended supergravity theories have long been seen as potential theories of quantum gravity although they were displaced by superstring theories [1] as favored candidates for a "final theory". The (quantum) M theory has 10 dimensional supergravities as low energy limits and might be thought of either as a (non-perturbative) quantum version of eleven dimensional supergravity (including its extended solutions) or as a non-perturbative completion of superstring theories. The spectacular perturbative ultra-violet finiteness of string theories indicates that strings provide a physical regulator of supergravities in 10 dimensions. In this paper we shall explore the ultra-violet behavior of $D = 10$, $N = 1$ supergravity and its dimensional descendants from a field theory viewpoint. In this we are attempting to "work-up" to the quantum theory of gravity rather than work down from string theory.

Of the extended supergravities two play a central role. Firstly, there is the $D = 11$, $N = 1$ maximally extended theory [2] which reduces to $N = 8$ in $D = 4$ [3]. In some ways this theory is the ultimate conventional point-particle field theory. The one-loop amplitude is potentially infinite for $D \geq 8$ although in dimensional regularisation it is only infinite for $D = 8$ (in the dimensional regularisation prescription one-loop amplitudes in odd dimensions are finite and the $D = 10$ infinity vanishes on-shell). We shall work entirely within the dimensional reduction scheme this being the most appropriate to study an anomaly-free supersymmetric theory.

At two-loops infinities have been calculated in the amplitudes for $D \geq 7$ [4] (including the $D = 11$ case. The eleven dimensional counterterm was subsequently evaluated in ref. [5]). The expectation is that this theory will be perturbatively infinite in four dimensions [6].

In this paper we shall examine features of the other interesting extended supergravity. This is the $D = 10$, $N = 1$ theory and its dimensional reduction descendants which include the $D = 4$, $N = 4$ supergravity theory (with a specific matter content). This supergravity is the gravitational sector of the low energy limit of both type-I and heterotic string theories [7]. For the one-loop amplitudes we may consider the $D = 10$, $N = 1$ supergravity together with arbitrary matter multiplets, although having less supersymmetry than the $D = 11$, $N = 1$ theory, it is not clear which is the most fundamental. A web of dualities relates the various string theories and if one introduces for instance the appropriate gauge group SO(32) in the type-I matter sector, this choice cancels the gravitational and gauge anomalies. When compactified to $D \leq 10$ type-I and type-II theories give rise to very special and similar symmetry groups. In fact it can be argued that these duality symmetries that have gained importance together with the realisation of the overwhelming necessity of non-perturbative effects are not that different between type-I and type-II supergravities. The tantalizing E_{10} is closely related to the hyperbolic Kac-Moody algebra obtained from the extrapolation of the Chamseddine-Sen sequence [8] namely the hyperbolic extension of D_8. In fact the symmetry groups of type I are fixed point sets under an involution of those of type II for all dimensions (at least three); this was mentioned in [9] and relies on Kac's description of the automorphisms of simple Lie algebras which can be found for instance in [10].

Clearly, the extended supergravities can only be UV finite if they possess symmetries which we do not fully understand yet, the implications of which will be at the non-perturbative level, or at least will require novel renormalisation techniques. Consequently, if these symmetries are operative at all, they should be approximately as powerful for type-I and -II theories. And one could infer from this that the structure of divergences should be somehow related in both theories which are two dual perturbative expansions. One should keep in mind though that the relation is nonperturbative.

The explicit (perturbative) calculation we shall perform is four graviton scattering at one-loop for $D \geq 4$. This will enable us to evaluate R^4 counterterms. For the one-loop amplitudes we may consider the $D = 10$, $N = 1$ supergravity multiplet with arbitrary matter, we shall call it type I slightly abusively. Although the theory will only be free from anomalies in $D = 10$ for special gauge groups, for $D = 8, 10$ the anomalies only manifest themselves in the five or six particle amplitudes. Furthermore the contribution is parity odd and finite whereas we shall consider divergences.

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Before calculating the counterterms, we shall calculate the entire amplitudes (including their finite part) and at first for a particular choice of external helicities — namely in the case where the external polarisation vectors are forced to be four dimensional. With this simplification it is possible to present the amplitude in a very elegant form as the sum over a few simple integral functions. From these amplitudes we can easily see the presence of ultra-violet infinities in $D = 8, 10$.

Unlike the situation in four dimensions, there are on-shell independent R^4 tensors in dimension $D \geq 8$, so in $D = 8$ the counterterms are not determined by single coefficients and it requires a full computation of the amplitude to fix their form. We find a beautiful factorisation of the infinite counterterms in the amplitude into a product of left times right kinematic factors in a manner very reminiscent of the relationships between gravity and Yang-Mills presented in ref. [4]. The factorisation can be understood from a string theory viewpoint but remains more obscure for field theorists. Offshell this factorisation is also not obvious at all from examining the field theory counterterms although the counterterms can be manipulated to reflect it.

2. D dimensional amplitudes with helicities in four dimensions

As a simple case, we first study amplitudes where the helicity of the external gravitons are restricted to lie in the four-dimensional space defined by the momenta. In this situation we can calculate the entire amplitude in a fairly simple form by breaking the amplitude into its helicity components. Even with the external helicities specified there are different contributions depending upon which supermultiplet is circulating in the internal loop. One may label the possible loop contributions according to the circulating four dimensional supermultiplets. The three contributions we shall distinguish are that from a $N = 8$ multiplet, that from a $N = 6$ matter multiplet and finally that from a $N = 4$ matter multiplet. In terms of these contributions the (pure) $N = 4$ supergravity one-loop amplitude is

$$M^{N=4} = M^{N=8} - 4M^{N=6,\mathrm{matter}} + 2M^{N=4,\mathrm{matter}} \,. \tag{2.1}$$

Also, of more interest to us is the "$N = 4^*$" one-loop amplitude for the dimensional reduction of $N = 1$, $D = 10$ supergravity to four dimensions which is

$$M^{N=4^*} = M^{N=8} - 4M^{N=6,\mathrm{matter}} + 8M^{N=4,\mathrm{matter}} \,. \tag{2.2}$$

It is often the case that nonmaximally supersymmetric theories do not remain irreducible upon dimensional reduction. If we consider the $N = 4^*$ (or type-I) supergravity coupled to $N = 4$ super-Yang-Mills matter the corresponding amplitude is

$$M^{N=4^*,G} = M^{N=8} - 4M^{N=6,\text{matter}} + (8 + \dim G)M^{N=4,\text{matter}}\,, \tag{2.3}$$

where $\dim G$ is the dimension of the gauge group. Let us call g the combination $g = (8 + \dim G)$. We have chosen to organize our amplitudes conveniently as linear combinations of the three supermultiplet contributions.

For $D = 10$ the three independent supersymmetric contributions are for instance the $N = 8$ and the $N = 4^*$ contributions as well as the "matter" $N = 4$ term. These have in turn reductions below 10 dimensions. If the external polarisations are forced to be four dimensional then the external gravitons have $\pm$ helicity. There are then only three independent "helicity amplitudes", $M(1^+, 2^+, 3^+, 4^+)$, $M(1^-, 2^+, 3^+, 4^+)$ and $M(1^-, 2^-, 3^+, 4^+)$. (We choose a convention where all particles are considered outgoing.) In any supersymmetric theory,

$$M(1^+, 2^+, 3^+, 4^+) = M(1^-, 2^+, 3^+, 4^+) = 0 \tag{2.4}$$

and the only non-zero independent amplitude is $M(1^-, 2^-, 3^+, 4^+)$. We have calculated our amplitudes using "String-based rules" [11, 12, 13, 14], (see appendix A) and verified the four-dimensional expressions using unitarity techniques. The details are presented in appendix B. The amplitudes are all regulated using dimensional reduction [15] with parameter $\epsilon = (D - D')/2$.

Calculating the $N = 8$ amplitude gives

$$M^{N=8}(1,2,3,4) = \left(\frac{\kappa}{2}\right)^2 [stuM^{\text{tree}}]\,(4\pi)^{-D/2} \times \left(I_4^D(s,t) + I_4^D(s,u) + I_4^D(t,u)\right), \tag{2.5}$$

where $I_4^D(s,t)$ denotes the D-dimensional scalar box integral function with ordering of legs 1234, s, t and u are the usual Mandelstam variables. The n-point scalar amplitude with external legs k_i defined by

$$\mathcal{I}_n^D = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2(p-k_1)^2 \cdots (p - \sum_{i=1}^{n-1} k_i)^2} \tag{2.6}$$

is related to the function I_n^D by

$$\mathcal{I}_n^D = i\frac{(-1)^n}{(4\pi)^{D/2}} I_n^D\,. \tag{2.7}$$

In general this is a function of the kinematic variables $(\sum_a^b k_i)^2$. Often we will express these variables which indicate the ordering of legs. For example $I_4^D(s,t)$ has ordering of legs 1234, $I_4^D(s,u)$ has ordering of legs 1243 etc... The $N = 8$ amplitude was originally evaluated using the low energy limit of string theory by Green, Schwarz and Brink [16]. Unlike the following two amplitudes it is valid for all external polarisations.

The remaining two amplitudes are more complicated and have the form

$$M^{N=6,\mathrm{matter}}(1^-,2^-,3^+,4^+) = -\left(\frac{\kappa}{2}\right)^2 [stuM^{\mathrm{tree}}](4\pi)^{-D/2}\Big(-\frac{1}{s}I_4^{D+2}(t,u) + \\ + \frac{(D-4)}{2s}\left(I_4^{D+2}(s,t)+I_4^{D+2}(s,u)-I_4^{D+2}(t,u)\right)\Big)\,, \tag{2.8}$$

$$M^{N=4,\mathrm{matter}}(1^-,2^-,3^+,4^+) = \left(\frac{\kappa}{2}\right)^2 [stuM^{\mathrm{tree}}](4\pi)^{-D/2}\Big(-\frac{1}{2s^2}\left(I_2^D(t)+I_2^D(u)\right) + \\ + \frac{(D(D+2))}{4s^2}I_4^{D+4}(u,t) + \frac{(2-D)(4-D)}{4s^2}I_4^{D+4}(s,t) + \\ + \frac{(2-D)(4-D)}{4s^2}I_4^{D+4}(s,u)\Big)\,. \tag{2.9}$$

Note the appearance of the D-dimensional scalar bubble integral, $I_2^D(t)$.

The tree amplitude for four-dimensional helicities is

$$M^{\mathrm{tree}}(1^-,2^-,3^+,4^+) = i\left(\frac{\kappa}{2}\right)^2 \frac{st}{u}\left(\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\right)^2. \tag{2.10}$$

where we have expressed the amplitude using a "spinor helicity" representation of the polarisation tensors. Spinor helicity techniques [17] were introduced for QCD calculation but by splitting the polarisation tensor

$$\epsilon^{\mu\nu} = \epsilon^{\mu}\bar{\epsilon}^{\nu} \tag{2.11}$$

they can be applied to gravity calculations also [18, 19].

These amplitudes are complete. They contain a great deal of information but in particular we can simply extract their ultra-violet infinities. Since we are using dimensional regularisation the one-loop integrals are only divergent in even dimensions. For a four point amplitude the expected counterterm is of the R^4 type which can only appear for $D \geq 8$ at one-loop.

For the $N=8$ amplitude the form of the amplitude is manifestly finite for $D<8$ since scalar box integrals are only divergent for $D \geq 8$. Extracting the divergences for $D=8$ and $D=10$ from the integrals yields

$$M^{N=8,D=8-2\epsilon} = \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^4}\,[stuM^{\mathrm{tree}}]\times\frac{1}{2\epsilon}\,, \\ M^{N=8,D=10-2\epsilon} = 0\,. \tag{2.12}$$

The $D=10$ one-loop amplitude is on-shell convergent within dimensional regularisation since the integrals give $\sim (s+t+u)=0$. A cut-off regularisation would give a non-zero result proportional to

$$\Lambda^2 \times stuM^{\mathrm{tree}}\,, \tag{2.13}$$

which leads to the well known [20, 21, 22, 23] counterterm $\Lambda^2 t_8 t_8 R^4$. (t_8 will be defined later.) Within dimensional regularisation the two-loop amplitude is the first divergence [4].

For the remaining two amplitudes we can evaluate the infinities in these expressions quite easily. The expressions are manifestly convergent for $D < 6$ for the $N = 6$ contribution and $D < 4$ for the $N = 4$ one. Extracting the divergences we find

$$
\begin{aligned}
M^{N=6,\mathrm{matter},D=6-2\epsilon}(1^-,2^-,3^+,4^+) &= 0\,,\\
M^{N=6,\mathrm{matter},D=8-2\epsilon}(1^-,2^-,3^+,4^+) &= \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^4}[stuM^{\mathrm{tree}}] \times -\frac{1}{24\epsilon}\,,\\
M^{N=6,\mathrm{matter},D=10-2\epsilon}(1^-,2^-,3^+,4^+) &= \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^5}[stuM^{\mathrm{tree}}] \times -\frac{s}{720\epsilon}\,,\\
M^{N=4,\mathrm{matter},D=6-2\epsilon}(1^-,2^-,3^+,4^+) &= 0\,,\\
M^{N=4,\mathrm{matter},D=8-2\epsilon}(1^-,2^-,3^+,4^+) &= \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^4}[stuM^{\mathrm{tree}}] \times \frac{1}{180\epsilon}\,,\\
M^{N=4,\mathrm{matter},D=10-2\epsilon}(1^-,2^-,3^+,4^+) &= \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^5}[stuM^{\mathrm{tree}}] \times \frac{s}{3360\epsilon}\,. \qquad (2.14)
\end{aligned}
$$

In both cases the $D = 6$ counterterm vanishes. In $D = 6$ the expected counterterm is R^3. However there is no supersymmetrisable R^3 counterterm and amplitudes contain no infinities. This is exactly the reason why the two-loop infinity vanishes in $D = 4$ supergravity.

Finally, recombining these poles to give the full physical amplitude with four dimensional helicities within type-I supergravity we find

$$
\begin{aligned}
M^{N=4^*}(1^-,2^-,3^+,4^+) = {} & \left(\frac{\kappa}{2}\right)^2 [stuM^{\mathrm{tree}}](4\pi)^{-D/2}\Big((I_4^D(s,t) + I_4^D(s,u) + I_4^D(t,u)) - \\
& - \frac{4}{s} I_4^{D+2}(t,u) \frac{2(4-D)}{s}\Big[I_4^{D+2}(s,t) - I_4^{D+2}(s,u) I_4^{D+2}(t,u))\Big] + \\
& + \frac{2(2-D)(4-D)}{s^2} I_4^{D+4}(s,t) + \frac{2(2-D)(4-D)}{s^2} I_4^{D+4}(s,u) + \\
& + \frac{2D(D+2))}{s^2} I_4^{D+4}(u,t) - \frac{4}{s^2}(I_2^D(t) + I_2^D(u))\Big)\,, \qquad (2.15)
\end{aligned}
$$

with divergences

$$
\begin{aligned}
M^{N=4^*,D=6}(1^-,2^-,3^+,4^+) &= 0\,,\\
M^{N=4^*,D=8}(1^-,2^-,3^+,4^+) &= \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^4}[stuM^{\mathrm{tree}}] \times \frac{32}{45\epsilon}\,,\\
M^{N=4^*,D=10}(1^-,2^-,3^+,4^+) &= \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^5}[stuM^{\mathrm{tree}}] \times \frac{s}{126\epsilon}\,. \qquad (2.16)
\end{aligned}
$$

If these are coupled to super-Yang-Mills the divergences become

$$M^{N=4^*,D=8}(1^-,2^-,3^+,4^+) = \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^4}[stuM^{\rm tree}] \times \frac{120+g}{180\epsilon}\,,$$
$$M^{N=4^*,D=10}(1^-,2^-,3^+,4^+) = \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^5}[stuM^{\rm tree}] \times \frac{(3g+56)s}{10080\epsilon}\,, \qquad (2.17)$$

where $g = (8 + \dim G)$.

Although these amplitudes are indicative of the divergences present they are not sufficient to determine completely the structure of the counterterms in higher dimensions. For example in $D = 8$ there are 7 independent R^4 tensors (on-shell) compared to 2 in $D = 4$. When restricting the external helicities and momenta to $D = 4$ naturally we loose most of the information. So to determine the exact counterterms we must calculate with arbitrary external helicity. This we shall do for $D = 8$, 10 in the following sections. We can also immediately see one interesting fact. For $N = 4$ the $D = 10$ divergence does not cancel — unlike in the $N = 8$ case. This implies that the counterterms must have a different R^4 structure. We shall explore this further also.

Let us conclude the present section with the remark that the strange coefficients appearing in the above formulas will actually be more reasonable looking in the higher dimensional amplitudes and result from the collapse of several invariants onto the same expression in lower dimension.

3. $D = 8$ counterterms

In this section we move on from the complete amplitudes and focus upon their infinity structure. We will relax our restriction to four dimensional helicities and obtain the infinity in the amplitudes in $D = 8$, 10 for arbitrary external polarisations. These amplitudes were calculated using the String-based method of ref. [11, 13, 14]. In this section we examine the $D = 8$ infinity structure. The dimensional reduction of $D = 10$, $N = 1$ to $D = 8$ is $D = 8$, $N = 1$ supergravity [24] plus matter.

In $D = 8$ the potential one-loop counterterm is an R^4 tensor. This is analogous to the situation in $D = 4$ at three loops where a potential R^4 term exists for $N = 1$ supergravity [6]. However, four dimensions is rather special because many of the potentially inequivalent R^4 tensors become equivalent at low dimensions. In fact, and as alluded to above, from the potential seven tensors on-shell (actually six remain after integration by parts) only two are inequivalent in four dimensions. Of these only one is compatible with supersymmetry — this is the well known Bel-Robinson tensor [25]. However for $D \geq 8$ all seven tensors are inequivalent and the structure of R^4 tensors is much richer. The supersymmetrisability has been discussed in [26].

Forgetting about supersymmetry we know from [27] that a general R^4 tensor in $D = 8$ is

$$a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4 + a_5T_5 + a_6T_6 + a_7T_7\,, \tag{3.1}$$

where

$$\begin{aligned}
T_1 &= (R_{p,q,r,s}R_{p,q,r,s})^2\,,\\
T_2 &= (R_{p,q,r,s}R_{p,q,r,t})(R_{p',q',r',s}R_{p',q',r',t})\,,\\
T_3 &= R_{p,q,r,s}R_{p,q,t,u}R_{t,u,v,w}R_{r,s,v,w}\,,\\
T_4 &= R_{p,q,r,s}R_{p,q,t,u}R_{r,t,v,w}R_{s,u,v,w}\,,\\
T_5 &= R_{p,q,r,s}R_{p,q,t,u}R_{r,v,t,w}R_{s,v,u,w}\,,\\
T_6 &= R_{p,q,r,s}R_{p,t,r,u}R_{t,v,u,w}R_{q,v,s,w}\,,\\
T_7 &= R_{p,q,r,s}R_{p,t,r,u}R_{t,v,q,w}R_{u,v,s,w}\,.
\end{aligned} \tag{3.2}$$

These are, on-shell, the independent tensors[1] and the combination

$$-\frac{T_1}{16} + T_2 - \frac{T_3}{8} - T_4 + 2T_5 - T_6 + 2T_7 \tag{3.3}$$

vanishes (or rather is a total divergence) being proportional to the Euler form.

$$E \sim \epsilon_{a_1a_2a_3a_4a_5a_6a_7a_8}\epsilon^{b_1b_2b_3b_4b_5b_6b_7b_8}R^{a_1a_2}{}_{b_1b_2}R^{a_3a_4}{}_{b_3b_4}R^{a_5a_6}{}_{b_5b_6}R^{a_7a_8}{}_{b_7b_8}\,. \tag{3.4}$$

In order to calculate the appropriate $N = 8$ counterterm we evaluate the (on-shell) amplitude and we find it factorises in the following way:

$$M^{N=8,D=8} = \frac{1}{\epsilon} \times \left(\frac{\kappa}{2}\right)^4 \frac{i}{(4\pi)^4}\,\frac{1}{2}K_1 \times K_1 \tag{3.5}$$

where

$$K_1 = tu(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4) + 2(\epsilon_1\cdot\epsilon_2)\Big(t(\epsilon_3\cdot k_1\epsilon_4\cdot k_2) + u(\epsilon_3\cdot k_2\epsilon_4\cdot k_1)\Big) + \cdots \tag{3.6}$$

where $\cdots$ denotes symmetrisation over the four indices 1234. The factorisation is easily understood if one regards $N = 8$ supergravity as the low energy limit of string theory, however it is much more obscure from a field theory viewpoint. In fact the tensor K_1 appears in string tree and loop amplitudes. (See ref. [1, eqs. (7.4.42) and (9.A.19)]). Its appearance in many diverse calculations is presumably due to the uniqueness of a tensor compatible with maximal supersymmetry.

The counterterm necessary to cancel this infinity is,

$$-\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{i}{(4\pi)^4}C^{N=8,D=8}\,, \tag{3.7}$$

[1]The Riemann tensor is undistinguished from the Weyl tensor in our R^4 terms for on-shell four point amplitudes.

where

$$C^{N=8,D=8} = a_2 \left[-\frac{T_1}{16} + T_2 - \frac{T_3}{8} - T_4 + 2T_5 - T_6 + 2T_7 \right] - \frac{1}{4} \left[T_4 - 4T_7 \right]. \tag{3.8}$$

There is one arbitrary coefficient a_2 since the Euler term vanishes on-shell. Choosing $a_2 = -1/4$ we have

$$C^{N=8,D=8} = -\frac{1}{4} \left[-\frac{T_1}{16} + T_2 - \frac{T_3}{8} - 0.T_4 + 2T_5 - T_6 - 2T_7 \right]. \tag{3.9}$$

This combination is precisely, the tensor combination

$$\frac{1}{128.6} t_8 t_8 R^4 \,, \tag{3.10}$$

which appears in the derivative expansion of the M-theory effective action [1, 22]. The tensor t_8 is defined in [1], here we drop its antisymmetric part. The reason why t_8 appears in trees as well as loops is connected to the form of the vertex operators and to triality.

Calculating with the (formal) ($N = 6, \text{matter}$) counterterm we find the infinity has the same tensor structure and is

$$M^{N=6,D=8} = -\frac{1}{12} M^{N=8,D=8} \,. \tag{3.11}$$

Finally, and more interestingly, we consider the infinity arising from the ($N = 4, \text{matter}$) multiplet. This also factorises into the form,

$$M^{N=4,D=8} = \frac{1}{\epsilon} \times \left(\frac{\kappa}{2} \right)^4 \frac{i}{(4\pi)^4} \times \frac{1}{720} K_1 \times K_2 \,, \tag{3.12}$$

where

$$\begin{aligned} K_2 = &-\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 (3t^2 + 5tu + 3u^2) - \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 (3s^2 + 5st + 3t^2) - \\ &- \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 (3s^2 + 5su + 3u^2) \\ &+ 2\epsilon_1 \cdot \epsilon_2 \Big(3s \epsilon_3 \cdot k_4 \epsilon_4 \cdot k_3 + t \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2 + u \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1 \Big) + \cdots \\ &- 12 (k_2 \cdot \epsilon_1 k_1 \cdot \epsilon_2 k_4 \cdot \epsilon_3 k_3 \cdot \epsilon_4 + k_3 \cdot \epsilon_1 k_4 \cdot \epsilon_2 k_1 \cdot \epsilon_3 k_2 \cdot \epsilon_4 + k_4 \cdot \epsilon_1 k_3 \cdot \epsilon_2 k_2 \cdot \epsilon_3 k_1 \cdot \epsilon_4) \,. \end{aligned} \tag{3.13}$$

We have organised K_2 according to the number of $\epsilon_i \cdot \epsilon_j$. The $\cdots$ denotes symmetrising the terms with a single $\epsilon_i \cdot \epsilon_j$. The counterterms necessary to cancel this are

$$-\frac{1}{\epsilon} \left(\frac{\kappa}{2} \right)^4 \frac{i}{(4\pi)^4} C^{N=4,D=8} \,, \tag{3.14}$$

where

$$C^{N=4,D=8} = \frac{1}{11520} \left(-3T_1 + 24T_2 - 6T_3 + 4T_4 + 0.T_5 + 0.T_6 + 32T_7 \right). \tag{3.15}$$

We can relate this also to specific tensors contracted against R^4. The tensor t_8 can be split into two pieces $t_{(12)}$ and $t_{(48)}$ each having the same symmetry properties as t_8. The tensors $t_{(12)}$ and $t_{(48)}$ contain 12 and 48 quartic monomials in the δ's respectively. They are the only two tensors which have the same symmetry properties of t_8 itself in eight dimensions [1]. Specifically

$$t_8 = \frac{1}{2}\Big(t_{(12)} + t_{(48)}\Big)\,, \tag{3.16}$$

where

$$\begin{aligned} t_{(12)}^{ijklmnpq} &= -\Big((\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})(\delta^{mp}\delta^{nq} - \delta^{mq}\delta^{np}) + (\delta^{km}\delta^{ln} - \delta^{kn}\delta^{lm})(\delta^{pi}\delta^{qj} - \delta^{pj}\delta^{qi}) + \\ &\qquad + (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})(\delta^{kp}\delta^{lq} - \delta^{kq}\delta^{lp})\Big)\,, \qquad (3.17) \\ t_{(48)}^{ijklmnpq} &= \Big(\delta^{jk}\delta^{lm}\delta^{np}\delta^{qi} + \delta^{jm}\delta^{nk}\delta^{lp}\delta^{qi} + \delta^{jm}\delta^{np}\delta^{qk}\delta^{li} + [i \leftrightarrow j] + [k \leftrightarrow l] + [m \leftrightarrow n]\Big)\,, \end{aligned}$$

where $[i \leftrightarrow j]$ denotes antisymmetrisation with respect to i and j. From these tensors we can define

$$\begin{aligned} A &= \frac{1}{4} t_{(12)} t_{(12)} \cdot R^4\,, \\ B &= \frac{1}{4} t_{(12)} t_{(48)} \cdot R^4\,, \\ C &= \frac{1}{4} t_{(48)} t_{(48)} \cdot R^4\,, \end{aligned} \tag{3.18}$$

where the $\cdot$ denotes the usual contraction of the upper and lower eight indices.

We can also express these tensor contractions as traces [21]

$$t_8 t_{(12)} \cdot R^4 = 48 t_8 \operatorname{Tr}(R^4)\,,, \qquad t_8 t_{(48)} \cdot R^4 = -12 t_8 \operatorname{Tr}(R^2)\operatorname{Tr}(R^2)\,. \tag{3.19}$$

In terms of these combinations the $N = 8$ counterterm of the type $t_8 t_8 R^4$ is just

$$C^{N=8,D=8} = \frac{1}{768}\Big(A + 2B + C\Big)\,. \tag{3.20}$$

The $N = 4$ counterterm can also be expressed in terms of A, B and C and is

$$C^{N=4,D=8} = \frac{1}{276480}\Big(-5A - 4B + C\Big)\,. \tag{3.21}$$

This second tensor structure has also played a role in the low energy limit of string theory, although in this case that of heterotic or type-I string theory. It has appeared in the ten dimensional effective action as seen by [32, 33, 34] both as a string tree and one-loop correction. Again the string formulation of tree amplitudes and thus of the appropriate invariant tensors gives an understanding of the factorisation of the amplitude which does not have a simple interpretation in field theory.

One checks easily that $5A + 4B - C$ contains a t_8 factor. The counterterm is also supersymmetrisable as shown in [26] where the most general lagrangeans of the form $R + R^4$ were considered (without vector multiplets).

Considering $N = 4^*$ supergravity coupled to matter we find

$$C^{N=4^*,D=8} = \frac{(g+480)}{2^{11}.3^3.5}\Big(A + 2B + C\Big) - \frac{6g}{2^{11}.3^3.5}\Big(A + B\Big)\,, \tag{3.22}$$

where $g = (8 + \dim G)$.

4. $D = 10$ counterterms

Ten dimensions is, of course, the natural home of both $N = 4^*$ supergravity and of superstring theory. However, within dimensional regularisation it is the $D = 8$ counterterms that match easily tensors which appear in string theory. In dimensional regularisation possible counterterms in D dimensions at L loops are

$$\partial^n R^m\,, \tag{4.1}$$

where $n+2m = (D-2)L+2$. With a cut-off regulator, string theory being a physical regulator, the equivalent terms are

$$\Lambda^n R^m\,. \tag{4.2}$$

For $D = 10$ our counterterms are thus of the form $\partial^2 R^4$. Indices have been suppressed in this expression and the full form is

$$T^{\alpha_1\alpha_2\mu_1\nu_1\rho_1\sigma_1\mu_2\nu_2\rho_2\sigma_2\mu_3\nu_3\rho_3\sigma_3\mu_4\nu_4\rho_4\sigma_4}\partial_{\alpha_1}R_{\mu_1\nu_1\rho_1\sigma_1}\partial_{\alpha_2}R_{\mu_2\nu_2\rho_2\sigma_2}R_{\mu_3\nu_3\rho_3\sigma_3}R_{\mu_4\nu_4\rho_4\sigma_4}\,. \tag{4.3}$$

Unless the tensor splits as follows

$$T^{\alpha_1\alpha_2\mu_1\nu_1\rho_1\sigma_1\mu_2\nu_2\rho_2\sigma_2\mu_3\nu_3\rho_3\sigma_3\mu_4\nu_4\rho_4\sigma_4} \sim \delta^{\alpha_1\alpha_2}T^{\mu_1\nu_1\rho_1\sigma_1\mu_2\nu_2\rho_2\sigma_2\mu_3\nu_3\rho_3\sigma_3\mu_4\nu_4\rho_4\sigma_4} \tag{4.4}$$

it will not be expressible in terms of the set of tensors found in $D = 8$ and those arising in string theory. As a matter of fact one of the counterterms we find does not split in this way.

Recall that in $D = 10$ the "physical" combinations are

$$\begin{aligned} &M^{N=8,D=10} \\ &M^{N=4,D=10} \\ &M^{N=4,D=10^*} = M^{N=8,D=10} - 4M^{N=6,D=10} + 8M^{N=4,D=10}\,. \end{aligned} \tag{4.5}$$

The results of calculating the infinities are firstly,

$$M^{N=8,D=10} = 0 \tag{4.6}$$

as expected. The infinities have two powers more of momentum as compared to $D = 8$. However we still find that all infinities factorise with one factor of K_1. The remaining factor L_i contains the extra two powers of momentum. Specifically we calculate

$$\begin{aligned} M^{N=4,D=10} &= \frac{1}{\epsilon} \times \left(\frac{\kappa}{2}\right)^4 \frac{-i}{(4\pi)^5} \frac{1}{60480} K_1 \times L_1 \\ M^{N=6,D=10} &= \frac{1}{\epsilon} \times \left(\frac{\kappa}{2}\right)^4 \frac{-i}{(4\pi)^5} \frac{1}{1440} K_1 \times L_2 \,, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} L_1 = {} & (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4) s(18u^2 + 41tu + 18t^2) + \cdots \\ & + 2(\epsilon_1 \cdot \epsilon_2)\Big(-t^2(18\epsilon_3 \cdot k_4 \epsilon_4 \cdot k_3 + \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2) - \\ & \qquad - u^2(18\epsilon_3 \cdot k_4 \epsilon_4 \cdot k_3 + \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1) - \\ & \qquad - tu(40\epsilon_3 \cdot k_4 \epsilon_4 \cdot k_3 + \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2 + \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1)\Big) + \cdots \\ & + 4\Big(t\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1 + 5t\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_2 + \\ & \quad + 6t\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_1 + t\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2 - \\ & \quad - 17t\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1 + 6t\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_2 - \\ & \quad - 18t\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1 - 18t\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_2 + \\ & \quad + t\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_1 - 4t\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2 - \\ & \quad - 18t\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1 - 23t\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2 - \\ & \quad - 17t\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_1 + 5t\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2 + \\ & \quad + 6t\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_1 - 18u\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_1 - \\ & \quad - 18u\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2 + 6u\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_1 - \\ & \quad - 23u\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1 - 17u\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_2 + \\ & \quad + 5u\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_1 + u\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2 + \\ & \quad + 5u\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1 + 6u\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_2 - \\ & \quad - 17u\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2 + u\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1 + \\ & \quad + 6u\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_2 - 4u\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1 + \\ & \quad + u\epsilon_1 \cdot k_3 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_2 - 18u\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2\Big) \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} L_2 = {} & -(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4) s(2u^2 + 3tu + 2t^2) + \cdots \\ & + 2(\epsilon_1 \cdot \epsilon_2)\Big(t^2(2\epsilon_4 \cdot k_3 \epsilon_3 \cdot k_4 + \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2) + \\ & \qquad + u^2(2\epsilon_3 4 \cdot k_4 \epsilon_4 \cdot k_3 + \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1) + \\ & \qquad + tu(2\epsilon_3 \cdot k_4 \epsilon_4 \cdot k_3 + \epsilon_3 \cdot k_1 \epsilon_4 \cdot k_2 + \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_1\Big) \cdots \end{aligned}$$

$$
\begin{aligned}
-4\Big(&-u\epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_2\epsilon_4\cdot k_1-u\epsilon_1\cdot k_3\epsilon_2\cdot k_3\epsilon_3\cdot k_1\epsilon_4\cdot k_2+\\
&+u\epsilon_1\cdot k_3\epsilon_2\cdot k_3\epsilon_3\cdot k_2\epsilon_4\cdot k_1-u\epsilon_1\cdot k_2\epsilon_2\cdot k_1\epsilon_3\cdot k_2\epsilon_4\cdot k_2-\\
&-u\epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_1\epsilon_4\cdot k_1-2u\epsilon_1\cdot k_2\epsilon_2\cdot k_1\epsilon_3\cdot k_1\epsilon_4\cdot k_1-\\
&-2u\epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_1\epsilon_4\cdot k_2+u\epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_2\epsilon_4\cdot k_2-\\
&-u\epsilon_1\cdot k_2\epsilon_2\cdot k_1\epsilon_3\cdot k_2\epsilon_4\cdot k_1+u\epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_1\epsilon_4\cdot k_2+\\
&+2u\epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_2\epsilon_4\cdot k_1-2u\epsilon_1\cdot k_2\epsilon_2\cdot k_1\epsilon_3\cdot k_1\epsilon_4\cdot k_2-\\
&-2t\epsilon_1\cdot k_2\epsilon_2\cdot k_1\epsilon_3\cdot k_2\epsilon_4\cdot k_1-2t\epsilon_1\cdot k_2\epsilon_2\cdot k_1\epsilon_3\cdot k_2\epsilon_4\cdot k_2+\\
&+t\epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_1\epsilon_4\cdot k_1+2t\epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_1\epsilon_4\cdot k_2-\\
&-2t\epsilon_1\cdot k_2\epsilon_2\cdot k_3\epsilon_3\cdot k_2\epsilon_4\cdot k_1-t\epsilon_1\cdot k_2\epsilon_2\cdot k_1\epsilon_3\cdot k_1\epsilon_4\cdot k_2-\\
&-t\epsilon_1\cdot k_2\epsilon_2\cdot k_1\epsilon_3\cdot k_1\epsilon_4\cdot k_1-t\epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_1\epsilon_4\cdot k_2+\\
&+t\epsilon_1\cdot k_3\epsilon_2\cdot k_3\epsilon_3\cdot k_1\epsilon_4\cdot k_2-t\epsilon_1\cdot k_3\epsilon_2\cdot k_3\epsilon_3\cdot k_2\epsilon_4\cdot k_1+\\
&+t\epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_2\epsilon_4\cdot k_1-t\epsilon_1\cdot k_3\epsilon_2\cdot k_1\epsilon_3\cdot k_2\epsilon_4\cdot k_2\Big)\,. \qquad (4.9)
\end{aligned}
$$

The infinities can be canceled by specific combinations of $\partial^2 R^4$ counterterms. Once more we find local Lorentz invariant counterterms to cancel the infinities. As a working hypothesis we assumed factorisability to make this tractable. The strategy is to calculate arbitrary on-shell scalars of the form $\partial^2 F^4$ times the previous $t_8 F^4$, to deduce from the resulting three parameter expression for the amplitude the values of these coefficients and finally to replace the two polynomial solutions in the F's by the corresponding invariants in the fourth order in the Riemann tensor. (On-Shell equivalent to the Weyl tensor at this order.) As is explained in detail in appendix C we may choose to express the counterterms in terms of the following set of tensors,

$$
\begin{aligned}
S_1 &= (\partial_\alpha R_{p,q,r,s}\partial_\alpha R_{p,q,r,s})(R_{p',q',r',s'}R_{p',q',r',s'})\\
S_2 &= (\partial_\alpha R_{p,q,r,s}\partial_\alpha R_{p,q,r,t})(R_{p',q',r',s}R_{p',q',r',t})\\
S_3 &= (\partial_\alpha R_{p,q,r,s}R_{p,q,r,t})(\partial_\alpha R_{p',q',r',s}R_{p',q',r',t})\\
S_4 &= \partial_\alpha R_{p,q,r,s}\partial^\alpha R_{p,q,t,u}R_{t,u,v,w}R_{r,s,v,w}\\
S_5 &= \partial_\alpha R_{p,q,r,s}\partial^\alpha R_{p,q,t,u}R_{r,t,v,w}R_{s,u,v,w}\\
S_6 &= \partial_\alpha R_{p,q,r,s}\partial^\alpha R_{p,q,t,u}R_{r,v,t,w}R_{s,v,u,w}\\
S_7 &= \partial_\alpha R_{p,q,r,s}\partial^\alpha R_{p,t,r,u}R_{t,v,u,w}R_{q,v,s,w}\\
S_8 &= \partial_\alpha R_{p,q,r,s}\partial^\alpha R_{p,t,r,u}R_{t,v,q,w}R_{u,v,s,w}\\
S_9 &= R_{m,b,c,d}R_{n,b,c,d}\partial_m R_{e,f,g,h}\partial_n R_{e,f,g,h}\\
S_{10} &= \partial_p R_{a,b,l,m}\partial_q R_{a,b,r,s}R_{q,d,l,r}R_{p,d,m,s}\,. \qquad (4.10)
\end{aligned}
$$

For S_1 to S_8 the derivatives are contracted with each other and these S_i's are related to derivatives acting upon the T_i's of the previous sections. Tensors S_9 and S_{10} however have the derivatives contracted into the Riemann tensors. Of course many

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tensors of the form $\partial^2 R^4$ vanish on-shell since they produce amplitudes of the form

$$\sim (s+t+u) \times \text{tensor} = 0 \,. \tag{4.11}$$

In terms of the S_i's the infinities are canceled by the counterterms

$$-\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{i}{(4\pi)^5} C^{N=6,D=10} \qquad \text{and} \qquad -\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{i}{(4\pi)^5} C^{N=4,D=10}\,, \tag{4.12}$$

where

$$\begin{aligned} C^{N=6,D=10} &= \frac{1}{4.720}\Big(S_1 - 12S_2 - 4S_3 + 2S_4 + 0.S_5 - 8S_6 + 16S_7 + 8S_8\Big) \\ C^{N=4,D=10} &= \frac{1}{4.6048}\Big(-9S_1 + 76S_2 - 44S_3 - 30S_4 + 56S_5 - \\ &\qquad - 88S_6 - 16S_7 + 88S_8 - 24S_9 + 95S_{10}\Big) \end{aligned} \tag{4.13}$$

It is far from obvious that such counterterms lead to infinities which factorise, however we can manipulate them to do so. In fact it is possible to express both tensors in the form

$$\begin{aligned} t_{10} t_8 \partial^2 R^2 \equiv\; & t_{10}^{a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}} t_{8\,b_1b_2b_3b_4b_5b_6b_7b_8} \times \\ & \times \partial_{a_1} R_{a_2a_3}{}^{b_1b_2} \partial_{a_4} R_{a_5a_6}{}^{b_3b_4} R_{a_7a_8}{}^{b_5b_6} R_{a_9a_{10}}{}^{b_7b_8}\,, \end{aligned} \tag{4.14}$$

where we have chosen to contract the derivatives into the t_{10} tensor.

The specific tensors are

$$\begin{aligned} t_{10}^{N=4,a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}} &= 10\delta^{a_1a_4}\delta^{a_2a_5}\delta^{a_3a_6}\delta^{a_7a_9}\delta^{a_8a_{10}} + 4\delta^{a_1a_4}\delta^{a_2a_{10}}\delta^{a_3a_5}\delta^{a_6a_7}\delta^{a_8a_9} + \\ &\quad + 4\delta^{a_1a_{10}}\delta^{a_2a_5}\delta^{a_3a_6}\delta^{a_4a_7}\delta^{a_8a_9}\,, \\ t_{10}^{N=6,a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}} &= \delta^{a_1a_4}\delta^{a_2a_5}\delta^{a_3a_6}\delta^{a_7a_9}\delta^{a_8a_{10}} - 4\delta^{a_1a_4}\delta^{a_2a_{10}}\delta^{a_3a_5}\delta^{a_6a_7}\delta^{a_8a_9}\,. \end{aligned} \tag{4.15}$$

Where possible one must antisymmetrise with respect to the pairs of indices $a_2 \leftrightarrow a_3$ etc and symmetrise with respect to pairs of couples of indices $(a_2a_3) \leftrightarrow (b_1b_2)$. The tensor for $N = 4^*$ and $N = 4^*$ coupled to matter being the appropriate linear combination of these. Both tensors define $\partial^2 R^4$ tensor which must be $D = 10, N = 1$ supersymmetrisable. Presumably the vanishing of the $M^{N=8,D=10}$ is because no $D = 10, N = 2$ supersymmetrisable $\partial^2 R^4$ tensor exists.

The counterterms for $D = 10, N = 1$ supergravities are given by linear combinations of the various $M^{D=10}$ and in fact the infinities are given by $K_1 \times \sum_i c_i L_i$ since the various infinities factorise in this way. At the level of the counterterms, all type-I supergravity divergences contain the factor t_8.

5. Conclusions

In this paper we have determined the one-loop ultra-violet behavior of the dimensional reductions of $D = 10$, $N = 1$ supergravity. We have examined this by calculations involving physical on-shell (four-point) amplitudes. We have found a variety of results; some in complete agreement with expectations but some not so obvious from the field theory viewpoint. As expected, the amplitudes are one-loop finite for $D < 8$. For $D \geq 8$ there is a richer structure. In $D = 8$ we find infinities which correspond to R^4 counterterms and we have completely determined the R^4 structure. In $D = 4$ the structure of R^4 terms is fairly simple; there are only two independent tensors of which one is compatible with supersymmetry — the well known "Bel-Robinson" combination [25]. For $D \geq 8$, there are seven independent tensors. For the $D = 8$, $N = 1$ theory supersymmetry is less restrictive than for $D = 8$, $N = 2$ and this allows a further R^4 counterterm, in fact we do find infinities belonging to a new tensor structure.

Within dimensional regularisation, $D = 10$ counterterms to a four-point infinity are of the form $\partial^2 R^4$. The vanishing of the $D = 10$, $N = 2$ one-loop infinity is presumably due to the non-existence of a supersymmetric combination of $\partial^2 R^4$ counterterms (which does not vanish on-shell.) For $D = 10, N = 1$ we do however find a $\partial^2 R^4$ counterterm which is non-zero.

An interesting feature of the amplitudes (so far checked up to 1-loop level) is *factorisation*. For the $D = 8$ counterterms one finds that the infinities in the amplitude factorise as $K_1 \times K_2$ where the K_a are combinations of the external ϵ_i and k_i. Similarly the $D = 10$ counterterms factorise as $K_1 \times L_2$. This factorisation is far from manifest when examining the counterterms however. Regarding the amplitudes as arising from the low energy limit of string theory, this factorisation is the remnant of the string factorisation into left and right moving amplitudes. However, it should be noted that the string factorisation occurs *within* the loop momentum integral whereas the factorisation of infinity occurs in the amplitude.

The factorisation is expected whenever the tree corrections generate the same invariants. However the factorisation of the amplitudes is unexpected from a field theory viewpoint and is hinting towards an alternate description of gravity theories as a product of two Yang-Mills theories — as suggested in [4, 19]. Although such a formalism is natural in string theory it might well exist in a purely field theoretic context.

The counterterm structures we find are related to terms appearing in various places in string theory. This is not surprising since string theory provides a physical regularisation of supergravity. Also String/M theory shares many symmetries with supergravity theory and hence counterterms/effective actions are subject to the same constraints. In the present work we have calculated purely within a field theory context. The structures we find are presumably inherent to any regulator of supergravity whether a string theory or a more conventional one.

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A. Calculations

We have calculated using the String-based techniques, originally developed for Yang-Mills calculations [11, 35, 12] and then applied to gravity calculations [13, 14]. Although originating in string theory the same formalism was subsequently shown to arise from a "world-line" approach to field theory [31]. We refer the reader to these papers for the details of the technique, here merely presenting the results for our amplitudes.

The techniques provide a formalism for obtaining the Feynman parameter integrals for on-shell amplitudes. The initial step is to draw all ϕ^3 diagrams, excluding tadpoles. There is no need to include diagrams with a loop isolated on an external leg since these vanish when dimensional regularisation is used. The external legs of these diagrams should be labeled, with diagrams containing all orderings included. The inner lines of trees attached to the loop are labeled according to the rule that as one moves from the outer lines to the inner ones, one labels the inner line with the same label as the most clockwise of the two outer lines attached to it. The contribution from each labeled n-point ϕ^3-like diagram with n_ℓ legs attached to the loop is

$$
\begin{aligned}
\mathcal{D} = {} & i\frac{(-\kappa)^n}{(4\pi)^{D/2}}\Gamma(n_\ell - D/2)\int_0^1 dx_{i_{n_\ell-1}} \int_0^{x_{i_{n_\ell-1}}} dx_{i_{n_\ell-2}} \cdots \int_0^{x_{i_3}} dx_{i_2} \int_0^{x_{i_2}} dx_{i_1} \times \\
& \times \frac{K_{\rm red}(x_{i_1},\ldots,x_{i_{n_\ell}})}{\left(\sum_{l<m}^{n_\ell} P_{i_l}P_{i_m} x_{i_m i_l}(1-x_{i_m i_l})\right)^{n_\ell - D/2}}\,,
\end{aligned}
\tag{A.1}
$$

where the ordering of the loop parameter integrals corresponds to the ordering of the n_ℓ lines attached to the loop, $x_{ij} \equiv x_i - x_j$. The x_{i_m} are related to ordinary Feynman parameters by $x_{i_m} = \sum_{j=1}^{m} a_j$. $K_{\rm red}$ is the "reduced kinematic factor", which the string-based rules efficiently yield in a compact form. The lines attached to the loop carry momenta P_i which will be off-shell if there is a tree attached to that line.

One obtains $K_{\rm red}$ by applying substitution rules to an overall kinematic factor,

$$
\begin{aligned}
\mathcal{K} = {} & \int \prod_{i=1}^{n} dx_i d\bar{x}_i \prod_{i<j}^{n} \exp\Big[k_i\cdot k_j G_B^{ij}\Big] \exp\Big[(k_i\cdot\epsilon_j - k_j\cdot\epsilon_i)\,\dot{G}_B^{ij} - \epsilon_i\cdot\epsilon_j\,\ddot{G}_B^{ij}\Big] \times \\
& \times \exp\Big[(k_i\cdot\bar{\epsilon}_j - k_j\cdot\bar{\epsilon}_i)\,\dot{\overline{G}}_B^{ij} - \bar{\epsilon}_i\cdot\bar{\epsilon}_j\,\ddot{\overline{G}}_B^{ij}\Big] \exp\Big[-(\epsilon_i\cdot\bar{\epsilon}_j + \epsilon_j\cdot\bar{\epsilon}_i)\,H_B^{ij}\Big]\Big|_{\rm multi-linear}\,,
\end{aligned}
\tag{A.2}
$$

where the 'multi-linear' indicates that only terms linear in all ϵ_i and $\bar{\epsilon}_i$ are included. The graviton polarization tensor is reconstructed by taking $\epsilon_i^\mu\bar{\epsilon}_i^\nu \to \epsilon_i^{\mu\nu}$. From a string theory perspective G_B is the bosonic Green function on the string world sheet, $\dot{G}_B$ and $\ddot{G}_B$ are derivatives of this Green function with respect to left-moving variables, and $\dot{\overline{G}}_B$ and $\ddot{\overline{G}}_B$ are derivatives with respect to right-moving ones. (Since a closed string is periodic the variables describing the string world sheet can be split into "left-moving" and "right-moving".) The term H_B^{ij} is the derivative of the Green function with respect to one left mover and one right mover variable. The functions G_B^{ij}, $\ddot{G}_B^{ij}$ and H_B^{ij} are to taken as symmetric in the i and j indices while $\dot{G}_B$ is antisymmetric. Although the above expression contains much information in string theory, when one takes the infinite string tension limit it should merely be regarded as a function which contains all the information necessary to generate $K_{\rm red}$ for all graphs. The utility of the string based method partially lies in this compact representation (which is valid for arbitrary numbers of legs!). The existence of an overall function which reduces to the Feynman parameter polynomial for each diagram is one of the most useful features of the String-based rules.

Slightly different substitutions rules are used depending upon particle type. The technique is particularly simple for supersymmetric theories where the cancellations between particle types circulating are manifest.

In general in Gravity theories, in a one-loop n-point amplitude the amplitude is a sum over diagrams with n_ℓ legs attached to the loop where $n_\ell \leq n$. For these diagrams the integrand is a polynomial of the Feynman parameters of degree $2n_\ell$. The overlying kinematic expression is of the form $\sum K_L \times K_R$. The substitutions act upon K_L and K_R independently — each being up to n_ℓ powers of Feynman Parameters. For a given particle circulating in the loop the substitution is of the form

$$K_{L/R} \to \pm S + C_p \,, \tag{A.3}$$

where S denotes a part common to all particle types and C_p is a "cycle" contribution which depends upon particle type. In general S is of degree n_ℓ whereas C_P is typically $n_\ell - 2$ or less. Bosonic/fermionic particles have a $\pm S$ term. In a supersymmetric theory we are thus guaranteed that the term $S \times S$ cancels and the Feynman parameter integral has maximum degree $n_\ell - 2$. For extended supersymmetry there can be further cancellations between the C_p terms and for a n-point amplitude the degree of the polynomial is $2n_\ell - 4$ for $N \geq 4$. For supersymmetry with $N \geq 4$ the cancellations imply that the only ϕ^3 diagrams contributing to a four point amplitude are the three box integrals. (This is not quite obvious the cancellations occur on K_L only [14] so if $n_\ell = 3$ for example the K_L becomes a polynomial of degree $3-4=-1$. In other words it cancels completely.)

In ref. [14] the low energy limit of string theory can be taken and the loop contributions from the different particle types disentangled to obtain the contributions due to a single graviton, Weyl fermion, scalar etc. Here we will be reconstructing the $N = 4^*$ amplitude again. A string theory consists of two sectors — the Neveu-

Schwarz sector (NS) and the Ramond sector (R). These sectors contribute in the low-energy limit

$$\begin{aligned} NS &\longrightarrow 8S + 2C_V \,, \\ R &\longrightarrow -8S - 8C_F \,. \end{aligned} \tag{A.4}$$

For a type-II superstring which has $N = 8$ supergravity as its low energy limit the contributions are of the form

$$(NS + R) \times (NS + R) \tag{A.5}$$

adds up to

$$4[C_V - 4C_F; C_V - 4C_F] \,. \tag{A.6}$$

For a superstring with $N = 4^*$ as its low energy limit the contribution is

$$(NS + R) \times NS^* \tag{A.7}$$

leading to a contribution

$$2[C_V - 4C_F; S] \,. \tag{A.8}$$

We use four dimensional helicity to simplify the overall kinematic expression whereas letting the particles circulate in any dimension $4 \leq D \leq 10$. This is very similar to the calculations of ref. [14] although here we let D vary from 4 to 10. The contributions we seek are for the $N = 8$, $N = 6$ and $N = 4$ matter multiplets.

For the amplitudes $M(1^-, 2^-, 3^+, 4^+)$ a choice of spinor helicity basis simplifies the kinematic expression enormously. Spinor Helicity techniques utilize a representation of the polarisation vectors in terms of spinor products,

$$\epsilon^{\pm}_{\mu}(p, k) = \pm \frac{\langle p \pm |\gamma_\mu| k \pm \rangle}{\sqrt{2} \langle k \mp | p \pm \rangle} \,, \tag{A.9}$$

where p^μ is the momentum of the external state and k^μ is a "reference momentum" satisfying $k^2 = 0$. (Different choices of k^μ correspond to different gauge choices for ϵ^μ. The advantage of this technique is that the k^μ's may be chosen to simplify the combinations $\epsilon_i \cdot \epsilon_j$ and $\epsilon_i \cdot k_j$ which appear in K_{red}. For the amplitude $M(1^-, 2^-, 3^+, 4^+)$ choosing $(k_i) = (p_4, p_4, p_1, p_1)$ then

$$\begin{aligned} k_4 \cdot \epsilon_1 &= 0 \,, & k_3 \cdot \epsilon_1 &= -k_2 \cdot \epsilon_1 \,, \\ k_4 \cdot \epsilon_2 &= 0 \,, & k_3 \cdot \epsilon_2 &= -k_1 \cdot \epsilon_2 \,, \\ k_1 \cdot \epsilon_3 &= 0 \,, & k_4 \cdot \epsilon_3 &= -k_2 \cdot \epsilon_3 \,, \\ k_1 \cdot \epsilon_4 &= 0 \,, & k_3 \cdot \epsilon_4 &= -k_2 \cdot \epsilon_4 \,, \end{aligned} \tag{A.10}$$

and

$$\begin{aligned} \epsilon_i \cdot \epsilon_j &= 0 \,, \qquad i, j \neq 2, 3 \,, \\ \epsilon_2 \cdot \epsilon_3 &= -\frac{2}{t} k_1 \cdot \epsilon_2 k_2 \cdot \epsilon_3 \,. \end{aligned} \tag{A.11}$$

With this simplification the entire kinematic expression is given by a kinematic factor

$$(\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 \epsilon_4 \cdot k_2)^2 \sim [stu M^{\rm tree}] \tag{A.12}$$

times a combination of Green's functions. This simplifies the calculations considerably.

Step 1. The $M^{N=8}$ amplitude. In the language of [14] we want the contribution of the form

$$4[C_V - 4C_F; C_V - 4C_F] \tag{A.13}$$

for the Green's functions. Each of the combinations C_V and C_F are quadratic in the Feynman parameter polynomials however for the combination $C_V - 4C_F$ two powers cancel leaving just a constant. For all three boxes this coefficient is simple as $C_V - 4C_F \to 1/2$ and we find the amplitude is just a sum over scalar box integrals with the overall factor $[stu M^{\rm tree}]$.

Step 2. The $M^{N=6}$. In the language of [14] we want the contribution of the form

$$-4[C_V - 4C_F; C_F]\,. \tag{A.14}$$

Applying the substitution rules we obtain,

$$\left[stu \frac{M^{\rm tree}}{2} \right] \times \left(I_{1234}[f_1(a_i)] + I_{1243}[f_2(a_i)] + I_{1324}[f_3(a_i)] \right), \tag{A.15}$$

where

$$\begin{aligned}
I_{1234}[f_1(a_i)] &= I_{1234}[a_3 - a_3^2 + a_1 a_3]\,, \\
I_{1243}[f_2(a_i)] &= I_{1243}[a_1 - a_1^2 + a_1 a_3 - a_1 a_2 + a_2 a_3]\,, \\
I_{1324}[f_3(a_i)] &= I_{1324}[a_1 - a_1^2 - a_1 a_3]\,.
\end{aligned} \tag{A.16}$$

Note that this is true in any dimension.

Reducing these integrals to scalar integrals by one's favorite technique (our is that of ref. [36]) we have

$$\begin{aligned}
I^D_{1234}[-a_1 + a_1^2 - a_1 a_3] &= \frac{2}{s} \times \left(2 - \frac{D}{2}\right) I^{D+2}_{1234} + \frac{2}{st} I_2^D(t) \\
I^D_{1234}[2a_1 a_3] &= \frac{2s + 2t(2 - D/2)}{su} I^{D+2}_{1234} - \frac{2}{su} I_2^D(s) + \frac{2}{su} I_2^D(t)\,,
\end{aligned} \tag{A.17}$$

which yields the amplitude

$$-\frac{1}{s} I^{D+2}_{1324} + \frac{D-4}{2s} \left(I^{D+2}_{1234} + I^{D+2}_{1243} - I^{D+2}_{1324} \right) \tag{A.18}$$

as has been observed before, [37], the amplitudes with less supersymmetry have the dimensions of the box integrals shifted relative to the maximal supersymmetric case.

Step 3. The $M^{N=4}$. In the language of [14] we want the contribution of the form

$$2[C_V - 4C_F; S]\,. \tag{A.19}$$

For the boxes we get

$$\begin{aligned}
&I_{1234}\left[-a_3^2(1-a_3)^2 - a_2a_3a_4(1-a_3) + 2a_3^2(a_1+a_2)(a_1+a_4) + \frac{a_3(a_2+a_4)}{4}\right]\\
&I_{1243}\left[(a_1+a_2)(a_3+a_4)(a_3a_1 - a_1a_4 - a_3a_2) + \frac{(a_1a_4+a_3a_2)}{4}\right]\\
&I_{1324}\left[a_3^2(-a_4(a_1+a_4) - a_2(a_1+a_2) + a_2a_4) - a_3a_2a_4 + \frac{a_3(a_2+a_4)}{4}\right].
\end{aligned}\tag{A.20}$$

Reducing these integrals to scalar integrals we have

$$\begin{aligned}
&\frac{(4-D)(2-D)}{4s^2}I^{D+4}(s,t) + \frac{(4-D)(2-D)}{s^2}I^{D+4}(s,u) +\\
&\qquad + \frac{(D+2)D}{4s^2}I^{D+4}(u,t) - \frac{1}{2s^2}\left(I_2^D(t) + I_2^D(u)\right).
\end{aligned}\tag{A.21}$$

Step 4. The $M^{N=4^*}$. Adding together the contributions to have

$$M^{N=4^*} = M^{N=8} - 4M^{N=6} + 8M^{N=4} \tag{A.22}$$

the combination is then just

$$4[C_V - 4C_F; 4S + C_V] \tag{A.23}$$

as one might deduce directly.

At this point we should discuss which string theory is relevant. In the original work on String-based rules the low energy limit of a heterotic string theory was taken to obtain the rules. Later it was realised that the same information could be encoded within a formalism where the fermionic sectors were dropped [28]. One of the challenges in developing this technique was to break the link between the string theory and the field theory limit allowing calculations in more general field theories such as non-supersymmetric QCD to be performed [29].

In our work we are focusing on $N = 4$ supergravity which is the low energy limit of heterotic string theory and as such the string based rules "recombine" and simplify. $N = 4$ supergravity is of course the field theory limit of two of the fundamental string theories namely the type-I theory of open and closed strings and the Heterotic string theory consisting of only closed strings. Both these string theories contain gauge groups in the low energy limit which we decouple. In the type-I theory the decoupling is extremely simple: the gauge particles are in the open string sector so the contribution to a four graviton scattering amplitude from the $N = 4^*$ multiplet will come from the torus and Klein bottle amplitudes only. For a heterotic theory one must drop the contributions arising from the gauge particles arising from the bosonic string.

B. Unitarity checks

In this section we show how the 4-graviton amplitude in $D = 4$ may be checked by combining unitarity and helicity techniques [38, 39, 40]. This approach provides an alternative method to obtain part of our results and at same time a strong check of their consistency.

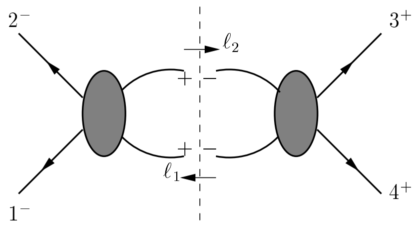

Figure 1: Helicity configuration for the $M(1^-,2^-,3^+,4^+)$ s-channel cut

Let us begin with the s-channel cut of the four-point amplitude $M(1^-,2^-,3^+,4^+)$ represented pictorially in figure 1. According to the Cutkosky rules, it is given by

$$\text{Disc } M^{1-\text{loop}}(1^-,2^-,3^+,4^+)\Big|_{\text{s}-\text{cut}} = i\int dLIPS \sum_{\substack{\text{internal}\\ \text{particles}}} M^{\text{tree}}(-\ell_1^+,1^-,2^-,\ell_2^+)\times \\ \times M^{\text{tree}}(-\ell_2^-,3^+,4^+,\ell_1^-)\,. \tag{B.1}$$

Here $dLIPS$ denotes the usual invariant Lorentz phase space and it contains an additional symmetry factor 1/2 to keep into account that 2 identical particles are going through the cut. Since we wish to construct the entire amplitude, we observe that we can replace the phase space integral by the cut of an unrestricted loop momentum integral

$$M^{1-\text{loop}}(1^-,2^-,3^+,4^+)\Big|_{\text{s}-\text{cut}} = \frac{1}{2}\int\frac{d^4\ell_1}{(2\pi)^4}\sum_{\substack{\text{internal}\\ \text{particles}}} \frac{i}{\ell_1^2}M^{\text{tree}}(-\ell_1^+,1^-,2^-,\ell_2^+)\times \\ \times\frac{i}{\ell_2^2}M^{\text{tree}}(-\ell_2^-,3^+,4^+,\ell_1^-)\Big|_{\substack{\text{s}-\text{cut}\\ \ell_1^2=\ell_2^2=0}}\,. \tag{B.2}$$

While eq. (B.1) includes only imaginary part, eq. (B.3) contains both real and imaginary parts. As indicated, eq. (B.3) holds only for those terms with s-channel branch cut; terms without an s-channel cut require a separate determination. A very useful feature of this identity is that we are free to use the on-shell conditions $\ell_1^2 = \ell_2^2 = 0$ to simplify the integrand: in fact only terms containing no cut in this channel would change.

The only internal particles, which give a non-vanishing result in the sum in eq. (B.3), are gravitons. Fermion contributions vanish because their helicity is not flipped by the graviton vertex implying that $M(g,g,\psi^+,\psi^+) = 0$. The same holds for vector and scalar amplitudes at tree level (taking particle/antiparticle instead of positive/negative helicity in the latter). (On a more formal ground, this follows from SUSY and chiral Ward identities [41].) Thus in our decomposition only the $N = 8$ supermultiplet will produce a non-vanishing s-cut, while the $N = 6$ and $N = 4$ supermultiplets will have only t/u-cuts. Let us focus on the former case.

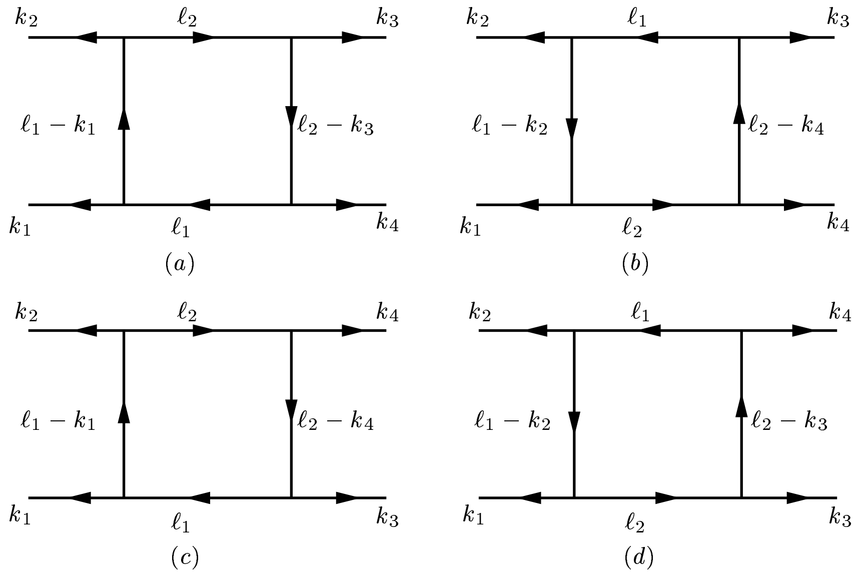

Figure 2: Boxes which contribute to the $M^{1-\text{loop}}(1^-,2^-,3^+,4^+)$ calculation. Note that (a) is related to (b) and (c) to (d) by redefinition of loop momenta.

In the standard helicity formalism (see e.g. [12]), the tree amplitude has the form

$$\begin{aligned} M^{\text{tree}}(1^-,2^-,3^+,4^+) &= i\frac{\kappa^2}{4}\left(\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\right)^2 \times \frac{st}{u} \\ &= -i\frac{\kappa^2}{4}\left[\frac{1}{t}+\frac{1}{u}\right]\times\left[s\frac{\langle 1\,2\rangle\,[34]}{[12]\,\langle 3\,4\rangle}\right]^2 . \end{aligned} \tag{B.3}$$

The equivalence between the two forms of the amplitude can be easily shown by exploiting the identities: $[ab]\langle ba\rangle = 2(a,b)$ and $[a_1a_2]\langle a_2a_3\rangle\cdots[a_{n-1}a_n]\langle a_n\ a_1\rangle = 1/2\,\mathrm{tr}[(1+\gamma_5)a_1\cdots a_n] \equiv \mathrm{tr}_+(a_1\cdots a_n)$. Inserting the r.h.s. of eq. (B.3) into the sum appearing in eq. (B.3), we obtain

$$\begin{aligned} -\frac{\kappa^4}{16}&\left[\frac{\langle 1\,2\rangle\,[\ell_1\ell_2]}{[12]\langle\ell_1\ell_2\rangle}s\right]^2\left[\frac{1}{(\ell_1-k_1)^2}+\frac{1}{(\ell_1-k_2)^2}\right]\times \\ &\times\left[\frac{[34]\langle\ell_1\ell_2\rangle}{\langle 3\,4\rangle\,[\ell_1\ell_2]}s\right]^2\left[\frac{1}{(\ell_2-k_3)^2}+\frac{1}{(\ell_2-k_4)^2}\right], \end{aligned} \tag{B.4}$$

which we can rearrange to be,

$$i\frac{\kappa^2}{4}stuM^{\text{tree}}(1^-,2^-,3^+,4^+)\left[\frac{1}{(\ell_1-k_1)^2}+\frac{1}{(\ell_1-k_2)^2}\right]\left[\frac{1}{(\ell_2-k_3)^2}+\frac{1}{(\ell_2-k_4)^2}\right], \tag{B.5}$$

where we have factorized out the kinematical factor given by the tree amplitude. Substituting this result into eq. (B.3) and expanding the products, we can interpret each term as coming from a box integral depicted in figure 2.

For example the product of the first two propagator leads to the integral

$$\int \frac{d^4 l_1}{(2\pi)^4} \frac{1}{\ell_1^2(\ell_1 - k_1)^2 \ell_2^2(\ell_2 - k_3)^2}\,, \tag{B.6}$$

which corresponds to the box integral (a) in figure 2.

$$\begin{aligned} M^{1-\text{loop}}\left(1^-, 2^-, 3^+, 4^+\right)\Big|_{\text{s}-\text{cut}} &= \left(\frac{\kappa}{2}\right)^2 \left[stuM^{\text{tree}}(1^-, 2^-, 3^+, 4^+)\right] \times \\ &\times (4\pi)^{-2} \left[I_4^4(s,t) + I_4^4(s,u)\right]\Big|_{\text{s}-\text{cut}} \end{aligned} \tag{B.7}$$

which is manifestly in agreement with the general result given in eq. (2.6) for the $N = 8$ contribution to the amplitude. ($I_4^4(t,u)$ is absent in (B.7) because it does not contain any s-cut. Its presence can be detected by looking at the t/u-cuts.)

The t- and u-channel cuts require essentially the same calculations since the helicity configuration are the same in both cases. We shall focus on the t-channel case and deduce the u-channel result from this by permuting $(1 \leftrightarrow 2)$. In this case the possible helicity configurations of the trees are those represented in figure 3. Since helicity is no longer flipped on either tree, the cut receives a non-vanishing contribution from each of the supermultiplet.

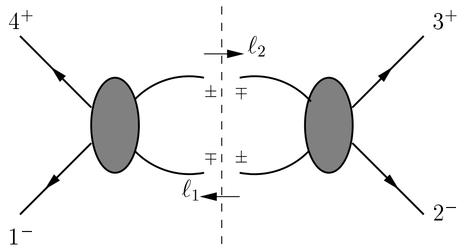

Figure 3: Helicity configurations for the $M(1^-, 2^-, 3^+, 4^+)$ t-channel cut

The contribution to the t-cut from states in the self-conjugate supermultiplet with N supersymmetries is given by

$$\begin{aligned} \text{Disc}M^{1-\text{loop}}\left(1^-, 2^-, 3^+, 4^+\right)\Big|_{\text{t}-\text{cut}} &= i\int dLIPS \sum_{\substack{J\in N \\ \text{supermultiplet}}} M^{\text{tree}}(4^+, 1^-, \ell_2^{J-}, -\ell_1^{J+}) \times \\ &\times M^{\text{tree}}(\ell_1^{J-}, -\ell_2^{J+}, 2^-, 3^+)\,. \end{aligned} \tag{B.8}$$

The above sum can be performed by recalling that all amplitudes are related, via supersymmetric Ward identities (see refs. [41]), to the scalar one according to the relations

$$\begin{aligned} M^{\text{tree}}(g^-, g^-, g^+, g^+) &= y^4 M^{\text{tree}}(g^-, \phi^-, \phi^+, g^+) \\ M^{\text{tree}}(g^-, \psi^-, \psi^+, g^+) &= y^3 M^{\text{tree}}(g^-, \phi^-, \phi^+, g^+) \\ M^{\text{tree}}(g^-, A^-, A^+, g^+) &= y^2 M^{\text{tree}}(g^-, \phi^-, \phi^+, g^+) \\ M^{\text{tree}}(g^-, \Lambda^-, \Lambda^+, g^+) &= y M^{\text{tree}}(g^-, \phi^-, \phi^+, g^+) \end{aligned} \tag{B.9}$$

where $y = \langle 1\,2\rangle / \langle 1\,3\rangle$ and g, ψ, A, Λ and ϕ denote the graviton, the gravitino, the vector, the spin 1/2 and the scalar respectively. Using, in fact, the relations (B.9) we can rearrange eq. (B.8) to be,

$$\begin{aligned}\mathrm{Disc}M^{1-\mathrm{loop}}(1^-,2^-,3^+,4^+)\Big|_{\mathrm{t-cut}} = i\int dLIPS\,\Big[M^{\mathrm{tree}}(4^+,1^-,\ell_2^{S-},-\ell_1^{S+})\times \\ \times M^{\mathrm{tree}}(\ell_1^{S-},-\ell_2^{S+},2^-,3^+)\rho_N\Big]\,.\end{aligned} \tag{B.10}$$

where $\rho_N = (x-x^{-1})^N$ with

$$x^2 = \frac{\langle 1\,\ell_2\rangle\,\langle 2\,\ell_1\rangle}{\langle 1\,\ell_1\rangle\,\langle 2\,\ell_2\rangle}\,. \tag{B.11}$$

The term ρ_N is a kinematical factor that encodes the sum over the different particles, while M^{tree} is, in this case, the amplitude for two scalars into two gravitons, whose explicit form can be obtained by combining eq. (B.9) with eq. (B.3).

If we restrict ourselves to choices with even $N = 2m$, we can simplify ρ_{2m} as follows

$$\begin{aligned}\rho_{2m} &= (x-x^{-1})^{2m} = \frac{(x^2-1)^{2m}}{x^{2m}} = \frac{(\langle 1\,\ell_2\rangle\,\langle 2\,\ell_1\rangle - \langle 1\,\ell_1\rangle\,\langle 2\,\ell_2\rangle)^{2m}}{(\langle 1\,\ell_2\rangle\,\langle 2\,\ell_1\rangle\,\langle 1\,\ell_1\rangle\,\langle 2\,\ell_2\rangle)^m} = \\ &= \frac{\langle 1\,2\rangle^{2m}\,\langle \ell_1\,\ell_2\rangle^{2m}}{(\langle 1\,\ell_2\rangle\,\langle 2\,\ell_1\rangle\,\langle 1\,\ell_1\rangle\,\langle 2\,\ell_2\rangle)^m} = \rho_8\left(\frac{\langle 1\,\ell_2\rangle\,\langle 2\,\ell_1\rangle\,\langle 1\,\ell_1\rangle\,\langle 2\,\ell_2\rangle}{\langle 1\,2\rangle^2\,\langle \ell_1\,\ell_2\rangle^2}\right)^{4-m} = \\ &= \left(\frac{\mathrm{tr}_+(4\ell_1 12\ell_1 3)}{t\,s^2}\right)^{4-m}\rho_8\,.\end{aligned} \tag{B.12}$$

where we have used the well-known identity $\langle ab\rangle\langle cd\rangle + \langle ac\rangle\langle db\rangle + \langle ad\rangle\langle bc\rangle = 0$.

It remains to compute the combination

$$\rho_8 M^{\mathrm{tree}}(4^+,1^-,\ell_2^{S-},-\ell_1^{S+}) \times M^{\mathrm{tree}}(\ell_1^{S-},-\ell_2^{S+},2^-,3^+)\,,$$

which corresponds to the contribution of the $N = 8$ self-conjugate supermultiplet to the t-cut. Inserting the explicit expression for the amplitudes and ρ_8 we find

$$-\left(\frac{\kappa^2}{4}\right)^2 t^2 \frac{\langle 1\,2\rangle^8}{\langle 2\,3\rangle^2\,\langle 4\,1\rangle^2}\frac{(k_1\cdot\ell_2)}{(k_1\cdot\ell_1)}\frac{(k_2\cdot\ell_2)}{(k_2\cdot\ell_1)}\frac{\langle \ell_1\,\ell_2\rangle^4}{(\langle 1\,\ell_2\rangle\,\langle \ell_1\,4\rangle\,\langle \ell_2\,2\rangle\,\langle 3\,\ell_1\rangle)^2}\,, \tag{B.13}$$

which in turn can be rewritten as

$$i\left(\frac{\kappa^2}{4}\right)\left[stuM^{\mathrm{tree}}(4^+,1^-,2^-,3^+)\right]\frac{t^2}{16(k_1\cdot\ell_1)(k_2\cdot\ell_1)(k_4\cdot\ell_1)(k_3\cdot\ell_1)}\,. \tag{B.14}$$

Here, as in the case of the s-cut, we have factorized out the tree amplitude and reduced everything to scalar products.

Therefore, we are led to write the following master-formula that encompasses all the possible contributions to the t-channel coming from a self-conjugate supermultiplet in $D = 4$

$$
\begin{aligned}
\mathrm{Disc}M^{1-\mathrm{loop}}(1^-,2^-,3^+,4^+)\,|_{\mathrm{t-cut}} = i\left(\frac{\kappa^2}{4}\right)\left[stuM^{\mathrm{tree}}(1^-,2^-,3^+,4^+)\right]\times \\
\times\, i\int dLIPS\frac{t^2}{16(k_1\cdot\ell_1)(k_2\cdot\ell_1)(k_4\cdot\ell_1)(k_3\cdot\ell_1)}\times \\
\times\left(\frac{\mathrm{tr}_+(4\ell_1 12\ell_1 3)}{t\,s^2}\right)^{4-m}. \qquad \text{(B.15)}
\end{aligned}
$$

The remarks made after eq. (B.3) are also valid here. In particular we are free to turn the scalar products appearing in the denominator of eq. (B.15) into propagators, when it is convenient.

The $N = 8$ contribution is selected by choosing $m = 4$. In this case, using the same procedure adopted for the s-channel, we can turn eq. (B.15) into a sum of boxes. We find a very similar result except that $s \leftrightarrow t$.

By posing $m = 3$, we are led to compute the $N = 6$ contribution. The first step is to expand the trace in the integrand and to simplify the common factors with the denominator

$$
\begin{aligned}
\frac{\mathrm{tr}_+(4\ell_1 12\ell_1 3)}{16(k_1\cdot\ell_1)(k_2\cdot\ell_1)(k_4\cdot\ell_1)(k_3\cdot\ell_1)} &= \frac{\mathrm{tr}(4\ell_1 12\ell_1 3)}{32(k_1\cdot\ell_1)(k_2\cdot\ell_1)(k_4\cdot\ell_1)(k_3\cdot\ell_1)} \qquad \text{(B.16)}\\
&= -\frac{1}{8}\left[t\left(\frac{1}{(\ell_1\cdot k_1)(\ell_1\cdot k_4)}+\frac{1}{(\ell_1\cdot k_2)(\ell_1\cdot k_3)}\right)-\right.\\
&\qquad\left.-\,u\left(\frac{1}{(\ell_1\cdot k_1)(\ell_1\cdot k_3)}+\frac{1}{(\ell_1\cdot k_2)(\ell_1\cdot k_4)}\right)\right]\\
&= -\frac{1}{8}\left[2\left(\frac{1}{(\ell_1\cdot k_1)}+\frac{1}{(\ell_1\cdot k_4)}-\frac{1}{(\ell_1\cdot k_2)}-\frac{1}{(\ell_1\cdot k_3)}\right)-\right.\\
&\qquad\left.-\,u\left(\frac{1}{(\ell_1\cdot k_1)(\ell_1\cdot k_3)}+\frac{1}{(\ell_1\cdot k_2)(\ell_1\cdot k_4)}\right)\right].
\end{aligned}
$$

By rewriting the denominators in the last line of eq. (B.16) as propagators,

$$
\begin{aligned}
\frac{1}{2}\left[\frac{1}{(\ell_1-k_1)^2}+\frac{1}{(\ell_1-k_4)^2}+\frac{1}{(\ell_1+k_2)^2}+\frac{1}{(\ell_1+k_3)^2}-\right.\\
\left.-\frac{u}{(\ell_1-k_1)^2(\ell_1+k_3)^2}-\frac{u}{(\ell_1-k_4)^2(\ell_1+k_2)^2}\right] \qquad \text{(B.17)}
\end{aligned}
$$

we can finally turn the phase space integral in eq. (B.15) into the t-cut of an unrestricted loop integral. As in the case of the s-channel, this is achieved through the substitution

$$
i\int dLIPS \to \int\frac{d\ell_1}{(2\pi)^4}\frac{i}{\ell_1^2}\frac{i}{\ell_2^2}\bigg|_{\mathrm{t-cut}}. \qquad \text{(B.18)}
$$

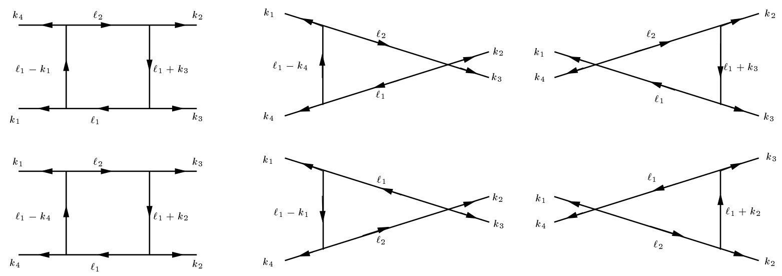

Figure 4: Boxes and triangles contributing to $M(1^-, 2^-, 3^+, 4^+)$ t-channel cut.

The final result is a combination of four triangles I_3^4 and two boxes I_4^4 and it is represented pictorially in figure 4.

In terms of the integral functions, we have

$$M^{1-\text{loop}}(1^-, 2^-, 3^+, 4^+)\Big|_{\text{t-cut}} = \left(-\frac{1}{2}\right)\left(\frac{\kappa}{2}\right)^2 [stuM^{\text{tree}}(1^-, 2^-, 3^+, 4^+)](4\pi)^{-2} \times \\ \times \frac{ut}{s^2}\left(\frac{2}{u}I_3^4(t) + \frac{2}{t}I_3^4(u) + I_4^4(t,u)\right)\Big|_{\text{t-cut}}, \qquad \text{(B.19)}$$

where we have already added the term $\frac{2}{t}I_3^4(u)$, whose presence can be detected by looking at the u-channel. Then using the identity

$$I_4^6(t,u) = -\frac{tu}{2s}\left(I_4^4(t,u) + \frac{2}{t}I_3^4(u) + \frac{2}{u}I_3^4(t)\right) \qquad \text{(B.20)}$$

we can recast the above combination of boxes and triangles integrals as a 6-dimensional box

$$M^{1-\text{loop}}(1^-, 2^-, 3^+, 4^+)\Big|_{\text{t-cut}} = \left(\frac{\kappa}{2}\right)^2 [stuM^{\text{tree}}(1^-, 2^-, 3^+, 4^+)](4\pi)^{-2}\frac{1}{s}I_4^6(t,u)\Big|_{\text{t-cut}}. \qquad \text{(B.21)}$$

This is exactly the result expected from formula (2.8).

The $N = 4$ contribution is more lengthy, but it can be computed along the same lines. We find again agreement with eq. (2.9). However in this last case some caution is in order: while the $N = 8$ and $N = 6$ amplitudes were cut-constructible, namely their exact one-loop expression could be deduced using unitarity [38, 39, 40], the $N = 4$ is not and only the cut expressions are guaranteed to coincide.

C. Form of the counterterms and factorisation of the divergences

The aim of the present appendix is to express each factor appearing in the $1/\epsilon$ poles of the $D = 8$ and $D = 10$ amplitudes in terms of Lorentz invariants in the product of four gauge field-strengths F_{ab} and their derivatives. One of the advantages of this analysis was to suggest a natural (and correct) form for the $\partial^2 R^4$ invariants entering the expression of our counterterm in $D = 10$. This would have been otherwise a hard task since the number of independent $\partial^2 R^4$ invariants in $D = 10$ dimensions which do not vanish on-shell is more than 30 [27]. At the end we shall briefly comment on a possible superstring interpretation of the observed factorization of the counterterms, which actually extends to the complete 1-loop amplitude as well.

It is useful to express the Riemann tensor in terms of the symmetrized product of two field-strengths:

$$R_{ab,cd} = \frac{1}{2}\Big(F_{ab}\bar{F}_{cd} + \bar{F}_{ab}F_{cd}\Big)$$
$$F_{ab} \equiv k_{[a}\,\epsilon_{b]}\,, \qquad \bar{F}_{ab} \equiv k_{[a}\,\bar{\epsilon}_{b]}$$

the vector polarizations ϵ_a and $\bar{\epsilon}_b$ being related to the graviton polarization ϵ_{ab} through a traceless symmetrized product: $\epsilon_{ab} = \epsilon_{(a}\bar{\epsilon}_{b)}$. As shown in sections 3 and 4, the $D = 8$ counterterms we find are of the form $K_1 \times K_j$ and similarly the $D = 10$ counterterms have the factorized form $K_1 \times L_j$ ($j = 1, 2$). As we shall show below, each of the factors can be expressed as a suitable invariant in the product of four vector field-strengths F_{ab} and their derivatives. In particular, in the $D = 8$ case we find that the factors K_i are two independent invariants constructed contracting suitable combinations of the tensors $t_{(12)}$ and $t_{(48)}$ with four gauge field strengths F_{ab}.

If we define:

$$\begin{aligned} N_1 &= -\frac{1}{12}\,t_{(12)} \cdot F^4 = (F_{ab}F^{ab})(F_{cd}F^{cd})\,, \\ N_2 &= \frac{1}{48}\,t_{(48)} \cdot F^4 = F_a{}^b F_b{}^c F_c{}^d F_d{}^a\,, \end{aligned} \tag{C.1}$$

we can express K_1 and K_2 as follows:

$$\begin{aligned} K_1 &= \frac{1}{16}\Big(N_1 - 4N_2\Big) = -\frac{1}{96}t_8 \cdot F^4\,, \\ K_2 &= -\frac{1}{16}\Big(5N_1 + 4N_2\Big)\,. \end{aligned} \tag{C.2}$$

In $D = 10$ we find that the factors L_i in the expression of the counterterm are related to invariants involving $\partial^2 F^4$ terms. There are only three independent such terms which do not vanish on-shell [30] which have the following form:

$$\begin{aligned} J_1 &= (\partial_m F_{ab}\partial^m F^{ab})(F_{cd}F^{cd})\,, \\ J_2 &= \partial_m F_a{}^b \partial^m F_b{}^c F_c{}^d F_d{}^a\,, \\ J_3 &= \partial_a F_m{}^n \partial_b F^m{}_n F_a{}^p F_b{}^p\,. \end{aligned} \tag{C.3}$$

In terms of the above invariants, L_1 and L_2 are expressed as follows:

$$\begin{aligned} L_1 &= \Big(5J_1 + 2J_2 - 2J_3\Big)\,, \\ L_2 &= \Big(-\frac{J_1}{2} + 2J_2\Big)\,. \end{aligned} \tag{C.4}$$

It is useful to express the product $K_1 \times J_k$ in the basis of the invariants S_I ($I = 1, \ldots, 10$):

$$\begin{aligned} K_1 \times J_1 &\longrightarrow -\frac{1}{8}\Big(-4S_1 + 32S_2 - 8S_4 + 16S_5\Big)\,, \\ K_1 \times J_2 &\longrightarrow -\frac{1}{8}\Big(-4S_2 - 4S_3 + 4S_5 - 8S_6 + 16S_7 + 8S_8\Big)\,, \\ K_1 \times J_3 &\longrightarrow -\frac{1}{8}\Big(-S_1 + 40S_3 + 10S_4 - 12S_5 + 80S_6 + 32S_7 - 80S_8 + 24S_9 - 95S_{10}\Big)\,. \end{aligned} \tag{C.5}$$

So that, multiplying by K_1 and using eqs. (A.4) we find the expression given in section 4 on the $D = 10$ counterterms ($N = 6$ and $N = 4$ respectively):

$$\begin{aligned} K_1 L_2 &\longrightarrow -\frac{1}{4}S_1 + 3S_2 + S_3 - \frac{S_4}{2} + 0.S_5 + 2S_6 - 4S_7 - 2S_8\,, \\ K_1 L_1 &\longrightarrow -\frac{1}{8}\Big(-18S_1 + 152S_2 - 88S_3 - 60S_4 + 112S_5 - \\ &\qquad - 176S_6 - 32S_7 + 176S_8 - 48S_9 + 190S_{10}\Big)\,. \end{aligned} \tag{C.6}$$

We found it useful also to express $K_1 \times J_3$ in a different basis of invariants M_l:

$$K_1 \times J_3 \longrightarrow 4M_1 - 16M_2 - 16M_3 + 4M_4 - 16M_5 + 4M_6\,, \tag{C.7}$$

where

$$\begin{aligned} M_1 &= \partial_a R_{eflm} \partial_d R_{efno} R_{dhno} R_{halm} \\ M_2 &= \partial_a R_{eflm} \partial_d R_{efno} R_{dhmo} R_{haln} \\ M_3 &= \partial_a R_{eflo} \partial_d R_{efno} R_{dhnm} R_{halm} \\ M_4 &= \partial_a R_{eflm} \partial_d R_{efno} R_{dhlm} R_{hano} \\ M_5 &= \partial_a R_{eflo} \partial_d R_{efno} R_{dhlm} R_{hanm} \\ M_6 &= \partial_a R_{efmo} \partial_d R_{efmo} R_{dhln} R_{haln} \end{aligned} \tag{C.8}$$

The expression of $K_1 \times J_3$ in terms of the tensors M_l is related through (torsion) Bianchi identities to its expression in terms of the S_i. The first two equations in (A.5) expressing $K_1 \times J_{1,2}$ in terms of S_I ($I = 1, \ldots, 8$) and (A.7) expressing $K_1 \times J_3$ in terms of the M_l ($l = 1, \ldots, 6$) can be derived directly by writing the three $K_1 \times J_i$ in the following more compact way:

$$\begin{aligned} K_1 \times J_i &= d_{J_i} t_8 \cdot \partial^2 R^4 \\ d_{J_i} t_8 \cdot \partial^2 R^4 &\equiv d_{J_i}^{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}} t_{8\, b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8} \times \\ &\quad \times \partial_{a_1} R_{a_2 a_3}{}^{b_1 b_2} \partial_{a_4} R_{a_5 a_6}{}^{b_3 b_4} R_{a_7 a_8}{}^{b_5 b_6} R_{a_9 a_{10}}{}^{b_7 b_8} \end{aligned}$$

and the 10-tensor d_{J_i} is defined as follows:

$$
\begin{aligned}
d_{J_1}^{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}} &= \delta^{a_1 a_4} \delta^{a_2 a_5} \delta^{a_3 a_6} \delta^{a_7 a_9} \delta^{a_8 a_{10}} \,, \\
d_{J_2}^{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}} &= \delta^{a_1 a_4} \delta^{a_2 a_{10}} \delta^{a_3 a_5} \delta^{a_6 a_7} \delta^{a_8 a_9} \,, \\
d_{J_3}^{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}} &= -\delta^{a_1 a_{10}} \delta^{a_2 a_5} \delta^{a_3 a_6} \delta^{a_4 a_7} \delta^{a_8 a_9} \,.
\end{aligned} \tag{C.9}
$$

The observed factorization of the $N = 1$ and $N = 2$ counterterms is consistent with the interpretation of these theories as low-energy limits of suitable closed superstring theories (heterotic and type-II respectively). Indeed, as it was shown above, each factor in the expression of the counterterms can be written in terms of invariants within suitable gauge theories. These gauge theories may be thought of as describing the low-energy excitations of the open string theories associated with the *left* (L) and *right* (R) movers of some closed string. From this point of view the factorization of the counterterms (and of the whole amplitudes) would seem quite natural: it is well known that the free Fock space of a closed string theory is the tensor product of the two open string Fock spaces corresponding to the (L) and (R) movers, moreover, retrieving the supergravity amplitudes from the α' expansion of closed string amplitudes (that is using superstring theory as a regulator of the effective supergravity theory), up to one-loop order the contributions from the two sectors are expected to factorise. However we wish to emphasize that the philosophy underlying our work is not to go from superstring theory "downwards", that is to study supergravity as its effective low energy theory, but on the contrary to move from field theory "upwards" and to study its effective action within a QFT framework, using dimensional reduction as a regularisation scheme, instead of string theory. From the field theory point of view the observed factorization is a highly non trivial result, consistent with the interpretation of the supergravities with 32 and 16 supercharges as low energy theories of type-II and heterotic superstring theories respectively. In particular the different forms of the counterterms in the maximal and non maximal cases may be related, in this perspective, to different supersymmetry constraints holding on the two sectors of the closed string theory. More specifically our results seem to suggest that in an open string sector the only possible invariants in four F_{ab} consistent with an $N = 1$ supersymmetry have to be constructed by saturating the indices of the field strengths with a t_8 tensor. This condition would restrict the possible invariants (non vanishing on-shell) in four F_{ab} and at most two derivatives, to just one possibility, namely $K_1 \propto t_8 \cdot F^4$. Indeed if we interpret the maximal supergravity as the low energy limit of type-II superstring, which has an $N = 1$ supersymmetry on both the (L) and (R) sectors, according to the above positions the only on-shell non vanishing counterterm is the one found in $D = 8$ (with no derivatives), which is indeed proportional to $K_1 \times K_1$. As far as the theory with 16 supercharges is concerned, it may be interpreted as the low energy effective theory of the heterotic superstring, which has an $N = 1$ supersymmetry on one sector (L) and an $N = 0$ on the other (R). Again, consistently with our results and the above considerations, we can associate the K_1 factor of the corresponding counterterms in $D = 8$ and $D = 10$ with the sector (L) constrained by supersymmetry.

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